

# On harmonic binomial series

Mark W. Coffey  
Department of Physics  
Colorado School of Mines  
Golden, CO 80401

(Received 2008)

April 29, 2008

## Abstract

We evaluate binomial series with harmonic number coefficients, providing recursion relations, integral representations, and several examples. The results are of interest to analytic number theory, the analysis of algorithms, and calculations of theoretical physics, as well as other applications.

## Key words and phrases

harmonic number, binomial coefficient, digamma function, polygamma function, generalized hypergeometric function, generalized harmonic number, Legendre function

## AMS classification numbers

05A10, 33C20, 33B15

## Introduction

The evaluation of harmonic number sums has been useful in analytic number theory for some time (e.g., [2, 3]). Recently, the evaluation of Euler sums [6, 10] has been important in various areas of theoretical physics, including in support of Feynman diagram calculations (e.g., [7]). More recently, the performance of generalized harmonic number sums has been useful in evaluating Feynman diagram contributions of perturbative quantum field theory [8]. In addition, harmonic number sums often arise in the analysis of algorithms. This especially applies to algorithms involving searching, sorting, or permutations (e.g., [14, 18]).

Here we are interested to evaluate sums containing simultaneously two types of special numbers of enumerative combinatorics: binomial coefficients and harmonic numbers. Our methods complement those of Refs. [5, 15]. The analytic approach of [5] is based upon hypergeometric summation and relies on identities of generalized hypergeometric functions  ${}_pF_p$  at the very special argument of 1. These identities include the Chu-Vandermonde formula for  ${}_2F_1$ , the Pfaff-Saalschütz theorem for  ${}_3F_2$ , the Dougall-Dixon theorem for  ${}_5F_4$ , and the Whipple transformation for  ${}_7F_6$ . In Ref. [15], symbolic computation using the Newton-Andrews-Zeilberger algorithm was used to discover harmonic number relations. These relations were motivated by considerations to prove certain 'supercongruences' for Apéry numbers.

Our results not only complement those of Ref. [5, 15], but demonstrate that various generalizations are possible, including the introduction of a summation parameter. The latter feature means that many previous results may be considered special cases,

and points out that many more results should be achievable in the future.

After first introducing some notation, we show how various recursion relations for the sums of interest may be developed directly. We then illustrate the development of integral representations for these sums, and provide examples. To emphasize the usefulness of the integral representations, we present low order cases explicitly. In addition, we give a representation of binomial-harmonic number sums in terms of the generalized hypergeometric function and its derivative. In particular, with regard to the Gauss hypergeometric function, we are then able to develop results in terms of Legendre function  $P_\nu$ .

In the final section, we introduce a variation. Namely, we consider binomial harmonic sums over the order of the generalized harmonic numbers. Additional considerations in analytic number theory motivate the study of these and related sums [9]. In particular, a certain combination of these sums represents a truncation of a dominating sum for the Li/Keiper constants.

We put  $H_n \equiv \sum_{k=1}^n 1/k$  for the usual harmonic numbers, and  $H_n^{(r)}$  for the generalized harmonic numbers

$$H_n^{(r)} \equiv \sum_{j=1}^n \frac{1}{j^r}, \quad H_n \equiv H_n^{(1)}. \quad (0.1)$$

These are given in terms of polygamma functions  $\psi^{(j)}$  as

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} \left[ \psi^{(r-1)}(n+1) - \psi^{(r-1)}(1) \right], \quad (0.2)$$

where  $\psi^{(r-1)}(1) = (-1)^r (r-1)! \zeta(r)$  and  $\zeta$  is the Riemann zeta function. The asymptotic form of  $H_n$  is well known,  $H_n = \ln n + \gamma + o(1)$ , where  $\gamma$  is the Euler constant.

Indeed, by Euler-Maclaurin summation we have

$$H_n = \ln n + \gamma + \frac{1}{2n} + \int_n^\infty \frac{P_1(x)}{x^2} dx, \quad (0.3)$$

where the periodic Bernoulli polynomial  $P_1(x) \equiv B_1(x - [x]) = x - [x] - 1/2$ . The asymptotic form of  $H_n^{(r)}$  for large  $n$  is immediately known from that of  $\psi^{(r-1)}(n+1)$ , and we have (e.g., [1], p. 260)

$$\psi^{(n)}(z) = (-1)^{n-1} \left[ \frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right) \right]. \quad (0.4)$$

We investigate here a subclass of generalized harmonic number sums

$$S_n^{(p)}(q, r, m, z) \equiv \sum_{j=0}^n j^p [H_j^{(q)}]^m \binom{n}{j}^r z^j, \quad |z| \leq 1. \quad (0.5)$$

In particular, in this paper we restrict for the most part to  $m = 1$ . When, in addition,  $q = r = 1$ , we write

$$S_n^{(p)}(z) \equiv \sum_{j=0}^n j^p H_j \binom{n}{j} z^j, \quad |z| \leq 1. \quad (0.6)$$

When simply  $p = 0$ , we omit the superscript.

We note a very convenient recursion for the general sums of Eq. (0.5):

$$S_n^{(p+1)}(q, r, m, z) = z \frac{\partial}{\partial z} S_n^{(p)}(q, r, m, z). \quad (0.7)$$

In this way, from an initial sum, successive sums may be obtained.

Although we do not follow this line of inquiry, we mention that many integral representations for binomial coefficients are known. These (e.g., [11], pp. 372-375;

[13], p. 53) include [11], (p. 375)

$$\binom{n}{m} = \frac{2^{n+2}}{\pi} \int_0^{\pi/2} \cos^n x \sin nx \sin 2mx \, dx = \frac{2^{n+2}}{\pi} \int_0^{\pi/2} \cos^n x \cos nx \cos 2mx \, dx. \quad (0.8)$$

As a contour integral, we have for complex  $\alpha$  and integral  $j \geq 1$ ,

$$\binom{\alpha}{j} = \frac{1}{2\pi i} \int_{|z|=r} (1+z)^\alpha z^{-j-1} dz, \quad 0 < r < 1. \quad (0.9)$$

We may note that this equation is particularly well formed for taking derivatives with respect to the parameter  $\alpha$ . If both  $n \geq 1$  and  $j \geq 1$  are integral, we no longer have a branch point at  $z = -1$ , and may write

$$\binom{n}{j} = \frac{1}{2\pi i} \int_{|z|=r} (1+z)^n z^{-j-1} dz, \quad 0 < r < \infty. \quad (0.10)$$

Equations (0.9) and (0.10) may be immediately verified by using the binomial theorem to compute the residue of the integrand at  $z = 0$ .

### Recursion relations for binomial-harmonic number sums

We have

**Proposition 1.** We have the recursion relation (a)

$$S_{n+1}(1) = 2S_n(1) + \frac{2^{n+1} - 1}{n+1}, \quad (1.1)$$

and (b)

$$S_{n+1}(z) = (1+z)S_n(z) + \frac{(z+1)^{n+1} - 1}{n+1}. \quad (1.2)$$

For part (a),  $S_1(1) = 1$ , and for part (b),  $S_1(z) = z$ .

**Proposition 2.** We have the recursion relation (a)

$$S_{n+1}^{(p)}(1) = 2S_n^{(p)}(1) + \sum_{\ell=0}^{p-1} \binom{p}{\ell} S_n^{(\ell)} + \frac{n}{2} \sum_{\ell=0}^p \binom{p}{\ell} {}_{\ell+1}F_{\ell}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1, 3; -1), \quad (1.3)$$

and (b)

$$S_{n+1}^{(p)}(z) = (1+z)S_n^{(p)}(z) + z \sum_{\ell=0}^{p-1} \binom{p}{\ell} S_n^{(\ell)} + \frac{n}{2} z^2 \sum_{\ell=0}^p \binom{p}{\ell} {}_{\ell+1}F_{\ell}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1, 3; -z). \quad (1.4)$$

For part (a),  $S_1^{(p)}(1) = 1$ , and for part (b),  $S_1^{(p)}(z) = z$ .

**Proposition 3.** We have the recursion relation (a)

$$S_n^{(1)}(z) = nzS_{n-1}(z) + (1+z)^n - 1. \quad (1.5)$$

Let

$$\beta_n(p, z) \equiv \sum_{j=1}^n j^{p-1} \binom{n-1}{j-1} \frac{z^j}{j} = z {}_{p-1}F_{p-2}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -z). \quad (1.6)$$

Then we have (b)

$$S_n^{(p)}(z) = nz \sum_{\ell=0}^{p-1} \binom{p-1}{\ell} S_{n-1}^{(\ell)}(z) + n\beta_n(p, z). \quad (1.7)$$

*Proof* of Proposition 1. The proof of part (b) subsumes that of part (a). By using a recursion relation for the binomial coefficient [1] (p. 822), we have

$$\begin{aligned} S_{n+1}(z) &= \sum_{j=0}^{n+1} H_j \binom{n+1}{j} z^j = \sum_{j=0}^{n+1} H_j \left[ \binom{n}{j} + \binom{n}{j-1} \right] z^j \\ &= S_n(z) + \sum_{j=1}^{n+1} H_j \binom{n}{j-1} z^j, \end{aligned} \quad (1.8)$$

where we have used the facts  $\binom{n}{n+1} = 0 = H_0$ . Further, by shifting the summation index and using the recursion relation for harmonic numbers, we have

$$\begin{aligned}
S_{n+1}(z) &= S_n(z) + \sum_{j=0}^n H_{j+1} \binom{n}{j} z^{j+1} \\
&= S_n(z) + \sum_{j=0}^n \left( H_j + \frac{1}{j+1} \right) \binom{n}{j} z^{j+1} \\
&= (1+z)S_n(z) + \frac{(z+1)^{n+1} - 1}{n+1},
\end{aligned} \tag{1.9}$$

wherein the latter sum may be obtained by integrating the binomial theorem.

*Proof* of Proposition 2a. For this part, omitting the  $z = 1$  argument, we have

$$\begin{aligned}
S_{n+1}^{(p)} &= \sum_{j=0}^{n+1} j^p H_j \binom{n+1}{j} = \sum_{j=0}^{n+1} j^p H_j \left[ \binom{n}{j} + \binom{n}{j-1} \right] \\
&= S_n^{(p)} + \sum_{j=1}^{n+1} j^p H_j \binom{n}{j-1} \\
&= S_n^{(p)} + \sum_{j=0}^n (j+1)^p H_{j+1} \binom{n}{j} \\
&= S_n^{(p)} + \sum_{j=0}^n \sum_{\ell=0}^p \binom{p}{\ell} j^\ell \left( H_j + \frac{1}{j+1} \right) \binom{n}{j}.
\end{aligned} \tag{1.10}$$

We next interchange the order of the two sums, and to complete this part of the Proposition we need to perform the sum

$$\sum_{j=1}^n \frac{j^\ell}{(j+1)} \binom{n}{j} = \sum_{j=0}^{n-1} \left[ \frac{(2)_j}{(1)_j} \right]^\ell \frac{1}{j+2} \binom{n}{j+1}, \tag{1.11}$$

where  $(a)_j = \Gamma(a+j)/\Gamma(a)$  is the Pochhammer symbol, and  $\Gamma$  is the Gamma function.

We further use  $(2)_j/(3)_j = 2/(j+2)$  and

$$\binom{n}{j+1} = \frac{(-1)^{j+1}}{(j+1)!} (-n)_{j+1} = \frac{(-1)^j}{(j+1)!} n(1-n)_j. \tag{1.12}$$

Appealing to the series definition of  ${}_{\ell+1}F_\ell$  completes part (a). The steps for part (b) are very similar, and are omitted.

*Proof of Proposition 3a.* We use the property  $j\binom{n}{j} = n\binom{n-1}{j-1}$ , so that

$$\begin{aligned}
S_n^{(1)}(z) &= n \sum_{j=1}^n \binom{n-1}{j-1} \left( \frac{1}{j} + H_{j-1} \right) z^j \\
&= n \left[ \sum_{j=1}^{n-1} \binom{n-1}{j-1} H_{j-1} z^j + H_{n-1} z^n + \sum_{j=1}^n \binom{n-1}{j-1} \frac{z^j}{j} \right] \\
&= n \left[ \sum_{j=1}^{n-2} \binom{n-1}{j} H_j z^{j+1} + H_{n-1} z^n + \sum_{j=1}^n \binom{n-1}{j-1} \frac{z^j}{j} \right] \\
&= n \left[ \sum_{j=1}^{n-1} \binom{n-1}{j} H_{j-1} z^{j+1} - H_{n-1} z^n + H_{n-1} z^n + \sum_{j=1}^n \binom{n-1}{j-1} \frac{z^j}{j} \right] \\
&= n \left[ z S_{n-1}(z) + \frac{1}{n} ((1+z)^n - 1) \right], \tag{1.13}
\end{aligned}$$

wherein the latter sum may be obtained by integrating the binomial theorem.

For part (b), we proceed similarly,

$$\begin{aligned}
S_n^{(p)}(z) &= \sum_{j=1}^n j^{p-1} j \binom{n}{j} H_j z^j \\
&= n \sum_{j=1}^n j^{p-1} \binom{n-1}{j-1} H_j z^j \\
&= n \left[ \sum_{j=1}^{n-1} \binom{n-1}{j-1} H_j z^j + n^{p-1} H_n z^n \right] \\
&= n \left[ \sum_{j=1}^n j^{p-1} \binom{n-1}{j-1} \left( \frac{1}{j} + H_{j-1} \right) z^j + n^{p-1} H_n z^n \right] \\
&= n \left[ \beta_n(p, z) + \sum_{j=0}^{n-1} (j+1)^{p-1} \binom{n-1}{j} H_j z^{j+1} \right]. \tag{1.14}
\end{aligned}$$



We then binomially expand, interchange sums, and the Proposition is complete.

*Remark.* Similar considerations can be given for quadratic and more generally nonlinear sums, that are outside of the present investigation.

### Examples

The recursion relations (1.1) and (1.2) with initial condition may be explicitly solved. For Eq. (1.1) we have

$$S_n(1) = 2^n H_n + \frac{1}{n+1} {}_2F_1(1, 1; n+2; -1) - 2^n \ln 2. \quad (1.15)$$

Since easily we have

$$\frac{1}{n+1} {}_2F_1(1, 1, n+2; -1) - 2^n \ln 2 = -2^n \sum_{j=1}^n \frac{1}{j2^j}, \quad (1.16)$$

we obtain

$$S_n(1) = 2^n \left( H_n - \sum_{j=1}^n \frac{1}{j2^j} \right), \quad (1.17)$$

a known result [15]. For Eq. (1.2) we find

$$S_n(z) = (1+z)^n \left[ H_n + \frac{1}{(z+1)^{n+1}} \frac{1}{(n+1)} {}_2F_1 \left( 1, n+1; n+2; \frac{1}{z+1} \right) + \ln \left( \frac{z}{z+1} \right) \right]. \quad (1.18)$$

By using a transformation formula [11] (p. 1043), we may write this as

$$S_n(z) = (1+z)^n \left[ H_n + \frac{1}{z(z+1)^n} \frac{1}{(n+1)} {}_2F_1 \left( 1, 1; n+2; -\frac{1}{z} \right) + \ln \left( \frac{z}{z+1} \right) \right]. \quad (1.19)$$

For  $z = 1$  here we have the explicit reduction to Eqs. (1.15) and (1.17). At  $z = -1$  we have

$${}_2F_1(1, 1; n+2; 1) = \frac{\Gamma(n+2)\Gamma(n)}{\Gamma^2(n+1)} = \frac{n+1}{n}. \quad (1.20)$$

Therefore, we obtain  $S_n(-1) = -1/n$ .

From Eqs. (1.5) and (1.18) we obtain

$$S_n^{(1)}(z) = nz(1+z)^{n-1} \left[ H_{n-1} + \frac{1}{(z+1)^n} \frac{1}{n} {}_2F_1 \left( 1, n; n+1; \frac{1}{z+1} \right) + \ln \left( \frac{z}{z+1} \right) \right]. \quad (1.21)$$

*Remark.* From the recursion relations of Proposition 2, it appears that the portions of the respective binomial-harmonic series containing harmonic numbers are given for  $p \geq 1$  by

$$\tilde{S}_n^{(p)}(1) = 2^{n-p} (n)_p \left[ H_{n-p} - \sum_{j=1}^{n-p} \frac{1}{j2^j} \right], \quad (1.22)$$

and

$$\tilde{S}_n^{(p)}(z) = (1+z)^{n-p} (n)_p \left[ H_{n-p} + \frac{1}{z(z+1)^{n-p}} \frac{1}{(n+1-p)} {}_2F_1 \left( 1, 1; n-p+2; -\frac{1}{z} \right) - \ln \left( \frac{z+1}{z} \right) \right]. \quad (1.23)$$

### Integral representations for linear binomial-harmonic number sums

We have

**Proposition 4.** We have the integral representation (a)

$$S_n^{(p)}(z) = nz \int_0^1 [t {}_pF_{p-1}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -zt) - {}_pF_{p-1}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -z)] \frac{dt}{t-1}, \quad (2.1)$$

and (b)

$$S_n^{(p)}(q, 1, 1, z) = \frac{(-1)^q}{(q-1)!} nz \int_0^1 [t {}_pF_{p-1}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -zt) - {}_pF_{p-1}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -z)] \frac{\ln^{q-1} t}{t-1} dt. \quad (2.2)$$

*Proof.* For part (a), we employ the relation  $H_j = \psi(j+1) + \gamma$ , where  $\psi = \Gamma'/\Gamma$  is the digamma function, together with an integral representation for this function [11] (p. 943):

$$\begin{aligned} S_n^{(p)}(z) &= \sum_{j=0}^n j^p [\psi(j+1) + \gamma] \binom{n}{j} z^j \\ &= \sum_{j=0}^n j^p \binom{n}{j} z^j \int_0^1 \frac{t^j - 1}{t - 1} dt. \end{aligned} \quad (2.3)$$

The integral is absolutely convergent and we may interchange summation and integration. We then rewrite the summation, using relation (1.12), and apply the series definition of  ${}_pF_{p-1}$ . Equation (2.1) follows.

For part (b), we use Eq. (0.2), writing

$$\begin{aligned} S_n^{(p)}(q, 1, 1z) &= \frac{(-1)^q}{(q-1)!} \sum_{j=0}^n j^p [\psi^{(q-1)}(j+1) - \psi^{(q-1)}(1)] \binom{n}{j} z^j \\ &= \frac{(-1)^q}{(q-1)!} \sum_{j=0}^n j^p \binom{n}{j} z^j \int_0^1 \left( \frac{t^j - 1}{t - 1} \right) \ln^{q-1} t \, dt. \end{aligned} \quad (2.4)$$

Here, we employed the result of multiply differentiating an integral representation for the digamma function to obtain that for the polygamma function. The integral is again absolutely convergent and we may interchange it with the summation. Carrying out the summation, we have Eq. (2.2).

*Remarks.* The integral representation of the polygamma function used above,

$$\psi^{(q-1)}(z) = \int_0^1 \left( \frac{t^{z-1} - 1}{t - 1} \right) \ln^{q-1} t \, dt = (-1)^q (q-1)! \zeta(q, z), \quad (2.5)$$

shows the connection with the Hurwitz zeta function  $\zeta(s, a)$  at integer values of  $q$ .

Reference [8] contains a number of other integral representations for harmonic numbers. Similarly, these may be applied to obtain other integral representations of the sums  $S_n^{(p)}$ .

The direct verification of the general property (0.7) from the integral representations requires the result

$$\begin{aligned} & \frac{\partial}{\partial z} {}_pF_{p-1}(2, 2, \dots, 2, 1-n; 1, 1, \dots, 1; -zt) \\ &= (n-1)2^{p-1}t {}_pF_{p-1}(3, 3, \dots, 3, 2-n; 2, 2, \dots, 2; -zt), \end{aligned} \quad (2.6)$$

as well as a transformation formula for  ${}_pF_{p-1}$ . It is possible that this transformation may be effected by noting

$$\left[ \frac{(3)_j}{(2)_j} \right]^{p-1} = \left( 1 + \frac{j}{2} \right)^{p-1} = \frac{1}{2^{p-1}} \left[ \frac{(2)_j}{(1)_j} + 1 \right]^{p-1}, \quad (2-n)_j = \left( 1 + \frac{j}{1-n} \right) (1-n)_j. \quad (2.7)$$

To show the utility of integral representations for binomial-harmonic number sums, we specialize in the following two sections to low order cases of the sums  $S_n^{(p)}$ , and provide corresponding details.

### Explicit expressions for sums $S_n^{(p)}(z)$

We have

**Proposition 5.** We have for  $|z| \leq 1$ ,

$$S_n(z) = nz(1+z)^{n-1} {}_3F_2\left(1, 1, 1-n; 2, 2; \frac{z}{1+z}\right), \quad (3.1)$$

giving

**Corollary 1.**

$$S_n^{(1)}(z) = (1+z)^{n-2} \left\{ 1 + z - \frac{1}{(1+z)^{n-1}} + n^2 z^2 {}_3F_2 \left( 1, 1, 1-n; 2, 2; \frac{z}{1+z} \right) \right\}, \quad (3.2a)$$

and

$$S_n^{(2)}(z) = z(1+z)^{n-3} \left\{ (1+z) \left[ 2n-1 + \frac{1-n}{(1+z)^n} \right] + n^2 z(1+nz) {}_3F_2 \left( 1, 1, 1-n; 2, 2; \frac{z}{1+z} \right) \right\}. \quad (3.2b)$$

The right side of Eq. (3.1) is easily verified to be a polynomial of degree  $n$  in  $z$ , as it must. For the (terminating)  ${}_3F_2$  function is a polynomial of degree  $n-1$  in  $z/(1+z)$ . When multiplied by the prefactor of  $z(1+z)^{n-1}$ , we obtain a polynomial of degree  $n$ . We could continue Corollary 1 indefinitely, but these instances serve our current purpose.

*Proof.* As with Eq. (2.3) at  $p = 0$ , we have

$$\begin{aligned} S_n(z) &= \sum_{j=0}^n \binom{n}{j} z^j \int_0^1 \frac{t^j - 1}{t-1} dt \\ &= - \int_0^1 [(1+z)^n - (1+zt)^n] \frac{dt}{t-1}. \end{aligned} \quad (3.3)$$

Changing variable with  $u = 1 - t$ , we have

$$S_n(z) = (1+z)^n \int_0^1 \left\{ 1 - \left[ 1 - \frac{zu}{1+z} \right]^n \right\} \frac{du}{u}, \quad (3.4)$$

and then obtain the Proposition.

Corollary 1 follows by applying property (0.7).

## Explicit expressions for some sums $S_n^{(p)}(1)$ in terms of harmonic numbers

We have

**Proposition 6.** We have

$$S_n(1) = 2^n \left[ H_n - \sum_{j=1}^n \frac{1}{j2^j} \right], \quad (4.1a)$$

$$S_n^{(1)}(1) = n2^{n-1} \left[ H_{n-1} - \sum_{j=1}^{n-1} \frac{1}{j2^j} \right] - 1 + 2^n, \quad (4.1b)$$

and

$$S_n^{(2)}(1) = 2^{n-2} \left\{ n(n+1) \left[ H_{n-2} - \sum_{j=1}^{n-2} \frac{1}{j2^j} \right] + [2n(1 - 2^{-n}) - 1] \frac{2n}{n-1} \right\}. \quad (4.1c)$$

Although Eqs. (4.1a) and (4.1b) are known results [15], our method of proof is different. In addition, the latter appears to be a rediscovery of earlier closed form summations [13].

*Proof.* Using the integral representation of the previous section, writing Eqs. (3.3) and (3.4) at  $z = 1$ , we have

$$\begin{aligned} S_n(1) &= \int_0^1 [-2^n - (1+t)^n] \frac{dt}{t-1} \\ &= 2^n \int_0^1 \left[ 1 - \left( 1 - \frac{u}{2} \right)^n \right] \frac{du}{u} \\ &= 2^n \left[ \psi(n+1) + \gamma + \int_{1/2}^1 \frac{(1-w)^n}{w} dw - \ln 2 \right] \\ &= 2^n \left[ H_n + \frac{1}{2^n(n+1)} {}_2F_1(1, 1, n+2; -1) - \ln 2 \right]. \end{aligned} \quad (4.2)$$

Here we have applied Eqs. (A5) and (A6) of Ref. [9]. By Eq. (1.16), Eq. (4.1a) follows.

Writing Eq. (2.3) at  $z = 1$  and  $p = 1$ , we have

$$\begin{aligned}
S_n^{(1)}(1) &= \sum_{j=0}^n j \binom{n}{j} z^j \int_0^1 \frac{t^j - 1}{t - 1} dt \\
&= n \int_0^1 [t(1+t)^{n-1} - 2^{n-1}] \frac{dt}{t-1} \\
&= n2^{n-1} \int_0^1 \left[ t \left( \frac{t+1}{2} \right)^{n-1} - 1 \right] \frac{dt}{t-1}.
\end{aligned} \tag{4.3}$$

Changing variable with  $u = 1 - t$ , we have

$$\begin{aligned}
S_n^{(1)}(1) &= n2^{n-1} \int_0^1 \left[ 1 - \left( 1 - \frac{u}{2} \right)^{n-1} \right] \frac{du}{u} - n2^{n-1} \int_0^1 \left( 1 - \frac{u}{2} \right)^{n-1} du \\
&= n2^{n-1} \left[ H_{n-1} - \sum_{j=1}^{n-1} \frac{1}{j2^j} \right] - 2^n(2^{-n} - 1),
\end{aligned} \tag{4.4}$$

giving Eq. (4.1b).

Following a similar procedure, writing Eq. (2.3) at  $z = 1$  and  $p = 2$ , we have

$$\begin{aligned}
S_n^{(2)}(1) &= \sum_{j=0}^n j^2 \binom{n}{j} z^j \int_0^1 \frac{t^j - 1}{t - 1} dt \\
&= n \int_0^1 [nt(1+nt)(1+t)^{n-2} - 2^{n-2}n(n+1)] \frac{dt}{t-1} \\
&= n2^{n-2} \int_0^1 \left[ nt(1+nt) \left( \frac{t+1}{2} \right)^{n-2} - n(n+1) \right] \frac{dt}{t-1}.
\end{aligned} \tag{4.5}$$

Now with  $u = 1 - t$ , we have

$$\begin{aligned}
S_n^{(2)}(1) &= 2^{n-2} \left\{ n(1-u)[(n+1) - nu] \left( 1 - \frac{u}{2} \right)^{n-2} - n(n+1) \right\} \frac{du}{u} \\
&= 2^{n-2} \left\{ n(n+1) \int_0^1 \left[ \left( 1 - \frac{u}{2} \right)^{n-2} - 1 \right] \frac{du}{u} + \int_0^1 \left( 1 - \frac{u}{2} \right)^{n-2} [-2n^2 - n + n^2u] du \right\}
\end{aligned}$$

$$\begin{aligned}
&= 2^{n-2} \left\{ n(n+1) \left[ H_{n-2} - \sum_{j=1}^{n-2} \frac{1}{j2^j} \right] \right. \\
&\quad \left. + n(1+2n) \frac{2}{n-1} (2^{1-n} - 1) + \frac{4n}{n-1} [1 - 2^{-n}(1+n)] \right\}. \tag{4.6}
\end{aligned}$$

Simplifying the latter terms on the right side of this equation gives Eq. (4.1c), and completes the Proposition.

### Hypergeometric approach

We have

**Proposition 7.** For integers  $p \geq 1$ ,

$$\begin{aligned}
S_n^{(0)} \left( 1, p, 1, \frac{1}{x} \right) &= -\frac{x^{-n}}{p} \frac{\partial}{\partial \nu} {}_pF_{p-1}[-\nu, \dots, -\nu; 1, \dots, 1; (-1)^p x] \Big|_{\nu=n} \\
&\quad + H_n x^{-n} {}_pF_{p-1}[-n, \dots, -n; 1, \dots, 1; (-1)^p x]. \tag{5.1}
\end{aligned}$$

*Proof.* We have

$$\binom{\nu}{j} = (-1)^j \frac{(-\nu)_j}{(1)_j}, \tag{5.2}$$

so that

$$\sum_{j=0}^{\infty} \binom{\nu}{j}^p x^j = {}_pF_{p-1}[-n, \dots, -n; 1, \dots, 1; (-1)^p x]. \tag{5.3}$$

Using

$$\frac{\partial}{\partial \nu} (-\nu)_j^p = p(-\nu)_j^p [\psi(\nu+1) - \psi(\nu-j+1)], \tag{5.4}$$

and the sum

$$\sum_{j=0}^n \binom{n}{j}^p H_{n-j} x^j = x^n \sum_{m=0}^n \binom{n}{m}^p H_m x^{-m} = x^n S_n^{(0)} \left( 1, p, 1, \frac{1}{x} \right), \tag{5.5}$$

we obtain the Proposition.



We focus on the  $p = 2$  case of Proposition 7, and obtain a number of series representations. For this, we introduce the Legendre functions of the first kind  $P_\nu$ , and the function

$$R_n(z) = \left. \frac{\partial P_\nu(z)}{\partial \nu} \right|_{\nu=n} - \ln\left(\frac{1+z}{2}\right) P_n(z), \quad (5.6)$$

where  $n \geq 0$  is an integer, and  $P_n$  is a Legendre polynomial. We have

**Proposition 8.**

$$\begin{aligned} S_n^{(0)}\left(1, 2, 1, \frac{1}{x}\right) &= -\frac{x^{-n}}{2} \frac{\partial}{\partial \nu} {}_2F_1(-\nu, -\nu; 1; x)|_{\nu=n} \\ &\quad + H_n x^{-n} {}_2F_1(-n, -n; 1; x). \\ &= -\frac{x^{-n}}{2} (1-x)^n R_n\left(\frac{1+x}{1-x}\right) + H_n x^{-n} (1-x)^n P_n\left(\frac{1+x}{1-x}\right). \end{aligned} \quad (5.7)$$

*Proof.* We first observe that through the use of a transformation formula for  ${}_2F_1$  we have

$$P_\nu(1-2x) = {}_2F_1(-\nu, \nu+1; 1; x) = (1-x)^\nu {}_2F_1\left(-\nu, -\nu; 1; \frac{x}{x-1}\right), \quad (5.8)$$

giving

$$P_\nu(z) = \left(\frac{z+1}{2}\right)^\nu {}_2F_1\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right). \quad (5.9)$$

Differentiation of this equation with respect to  $\nu$  and comparison with the defining relation (5.6) yields

$$R_n(z) = \left. \left(\frac{z+1}{2}\right)^n \frac{\partial}{\partial \nu} {}_2F_1\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \right|_{\nu=n}. \quad (5.10)$$

With a change of variable, we obtain Eq. (5.7).

We now discuss and illustrate Proposition 8, in the course of which we obtain

**Corollary 2**

$$S_n^{(0)}(1, 2, 1, 1) = \sum_{j=0}^n H_j \binom{n}{j}^2 = (2H_n - H_{2n}) \binom{2n}{n}, \quad (5.11)$$

recovering a known result [15].

There are several series representations of the function  $R_n$  available, including the Bromwich forms [4]

$$R_n(z) = \sum_{k=1}^n \frac{1}{k} [P_k(z) - P_{k-1}(z)] P_{n-k}(z), \quad (5.12)$$

$$R_n(z) = 2 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k+1}{(n-k)(n+k+1)} [P_k(z) - P_n(z)], \quad (5.13)$$

and from a formula of Jolliffe [12],

$$R_n(z) = -2 \ln \left( \frac{1+z}{2} \right) P_n(z) + \frac{1}{2^{n-1} n!} \frac{d^n}{dz^n} \left[ (z-1)^n (z+1)^n \ln \left( \frac{z+1}{2} \right) \right]. \quad (5.14)$$

Besides these older results,  $R_n$  and closely related polynomials have been very recently restudied [17]. Since the Legendre polynomial has  $P_n(1) = 1$  and  $P_k(-1) = (-1)^k$ , we readily see that the function  $R_n$  satisfies  $R_n(1) = 0$  and  $R_n(-1) = 2(-1)^n H_n$ . It is evident from the Bromwich formula (5.13) that

$$R_n(z) = 2(H_{2n} - H_n)P_n(z) + 2 \sum_{k=0}^{n-1} (-1)^{n+k} \frac{2k+1}{(n-k)(n+k+1)} P_k(z). \quad (5.15)$$

The property  $R_n(-1) = 2(-1)^n H_n$  follows immediately from this equation. With the use of the duplication formula satisfied by the digamma function, the property  $R_n(1) = 0$  may also be deduced from Eq. (5.15).

Equivalent to the Chu-Vandermonde summation  ${}_2F_1(-n, -n; 1; 1) = \binom{2n}{n}$ , is the relation, via Eq. (5.9),

$$\lim_{x \rightarrow 1} (1-x)^n P_n \left( \frac{1+x}{1-x} \right) = \binom{2n}{n}. \quad (5.16)$$

In addition, with the help of Eq. (5.15) and the use of partial fractions, we have

$$\lim_{x \rightarrow 1} (1-x)^n R_n \left( \frac{1+x}{1-x} \right) = -2 \binom{2n}{n} (H_{2n} - H_n). \quad (5.17)$$

Therefore, from Proposition 8 we obtain Corollary 2.

By the same method of this section, on taking more derivatives with respect to  $\nu$  in Eq. (5.1), it is possible to get higher values of  $q$  in the sums defined in Eq. (0.5). In this regard, we very briefly mention the functions

$$Q(x, z) = {}_2F_1(x, -x; 1; z) = \sum_{j=0}^{\infty} \frac{(x)_j (-x)_j}{(j!)^2} z^j, \quad Q(x) \equiv Q(x, -1), \quad (5.18)$$

and the expansion

$$Q(x) = 1 + \sum_{j=1}^{\infty} A_{2j} x^{2j}, \quad (5.19)$$

written in Ref. [16]. Obviously the Maclaurin coefficients here are given by

$$A_{2j} = \frac{1}{(2j)!} \left( \frac{d}{dx} \right)^{2j} Q(x) \Big|_{x=0}. \quad (5.20)$$

Then by the property (5.4) at  $p = 1$ , the coefficients  $A_{2j}$  involve sums and products of generalized harmonic numbers. Further connections with Legendre functions and Stirling numbers of the first kind we do not pursue here.

Among the generalized hypergeometric functions that reduce to polynomials is the family

$$H_n(x, a, z) = {}_3F_2(-n, n+1, x; 1, a; z), \quad (5.21)$$

where  $n \geq 0$  is an integer and  $x, a \in C$ , with  $a \neq -n-1, -n-2, \dots$ . These functions may be obtained by a certain integration over shifted Legendre polynomials, and similarly for higher order functions  ${}_pF_{p-1}$ . Obviously the set of functions of Eq. (5.21) includes  $H_n(n+1, 1, z)$  and related functions. One may then ask for a generalization of Proposition 8 for the case of  $p = 3$  in Proposition 7, an effort that we are leaving to future work.

We also mention an integral representation for the sum of Proposition 8, and how the case of Corollary 2 may be otherwise obtained. We have

$$S_n^{(0)}(1, 2, 1, z) = \int_0^1 [{}_2F_1(-n, -n, 1, zt) - {}_2F_1(-n, -n, 1, z)] \frac{dt}{t-1}. \quad (5.22)$$

As special case, we have

$$S_n^{(0)}(1, 2, 1, 1) = \binom{2n}{n} \int_0^1 \left[ \frac{1}{\binom{2n}{n}} {}_2F_1(-n, -n, 1, t) - 1 \right] \frac{dt}{t-1} \quad (5.23a)$$

$$= -\binom{2n}{n} \int_0^\infty \left[ \frac{1}{\binom{2n}{n}} (1-x)^{-n} P_n(1-2x) - 1 \right] \frac{dx}{x-1} \quad (5.23b)$$

$$= -\binom{2n}{n} \int_1^\infty \left[ \frac{1}{\binom{2n}{n}} \left( \frac{2}{1+z} \right)^n P_n(z) - 1 \right] \frac{dz}{1+z}, \quad (5.23c)$$

where we changed variable and used Eqs. (5.8) and (5.9). One way to demonstrate equality with the result of Corollary 2 is to show that the integral of this equation satisfies the same recursion relation  $Q_{n+1} - Q_n = 1/(2n+1) + 1/(2n+2) - 2/(n+1)$  as the quantity  $Q_n = H_{2n} - 2H_n$ , with initial condition  $Q_1 = -1/2$ . We also note the integral representation

$$Q_n = -\int_0^1 \frac{(t^n - 1)^2}{t-1} dt. \quad (5.24)$$

### Other class of binomial harmonic sums

Finally, we describe another class of sums, where the summation is now over the order of the generalized harmonic numbers,

$$\mathcal{S}_n(M, z) \equiv \sum_{m=2}^n \binom{n}{m} H_M^{(m)} z^m. \quad (6.1)$$

We have

**Proposition 9.** Let  $L_n^\alpha$  be the associated Laguerre polynomial of degree  $n$  (e.g., [1, 9, 11]). We have

$$\mathcal{S}_n(M, z) = z \int_0^1 \left[ L_{n-1}^1(z \ln t) - n \right] \left( \frac{t^M - 1}{t - 1} \right) dt. \quad (6.2)$$

*Proof.* We use relation (0.2) together with an integral representation for the polygamma function,

$$\begin{aligned} \mathcal{S}_n(M, z) &= \sum_{m=2}^n \binom{n}{m} z^m \frac{(-1)^{m-1}}{(m-1)!} \left[ \psi^{(m-1)}(M+1) - \psi^{(m-1)}(1) \right] \\ &= \sum_{m=2}^n \binom{n}{m} z^m \frac{(-1)^{m-1}}{(m-1)!} \int_0^1 \left( \frac{t^M - 1}{t - 1} \right) \ln^{m-1} t \, dt. \end{aligned} \quad (6.3)$$

Interchanging the finite summation with the integration and applying the power series definition of  $L_{n-1}^1$ , we obtain the Proposition.

*Remarks.* In the limit as  $M \rightarrow \infty$  in Eq. (6.2), we have the representation

$$\lim_{M \rightarrow \infty} \mathcal{S}_n(M, z) = -z \int_0^1 \left[ L_{n-1}^1(z \ln t) - n \right] \frac{dt}{t - 1}. \quad (6.4)$$

This is the same limit in which  $H_M^{(m)} \rightarrow \zeta(m)$ . In particular, we put

$$S_1(n) = \lim_{M \rightarrow \infty} [\mathcal{S}_n(M, -1) - \mathcal{S}_n(M, -1/2)]. \quad (6.5)$$

With a change of variable, we obtain agreement with the integral representation of  $S_1(n)$  of Ref. [9] (p. 216). In particular, this alternating binomial sum provides the apparently dominant contribution to the Li/Keiper constants  $\lambda_n$  of the Li criterion for the Riemann hypothesis [9]. This sum has been shown to be  $O(n \ln n)$ , describing the significant cancellation within the summand.

**Proposition 10.** We have

$$\mathcal{S}_n(M, z) = (1 + z)^n + \sum_{j=2}^M \left(1 + \frac{z}{j}\right)^n - nzH_M - M. \quad (6.6)$$

*Proof.* By using the recursion relation of generalized harmonic numbers, we have from the definition (6.1)

$$\mathcal{S}_n(M, z) = \mathcal{S}_n(M - 1, z) - 1 - \frac{n}{M}z + \left(1 + \frac{z}{M}\right)^n. \quad (6.7)$$

This recursion may also be found from the integral representation of Proposition 9.

We form the difference

$$\mathcal{S}_n(M, z) - \mathcal{S}_n(M - 1, z) = z \int_0^1 \left[ L_{n-1}^1(z \ln t) - n \right] t^{M-1} dt. \quad (6.8)$$

We then integrate by parts,

$$\begin{aligned} \mathcal{S}_n(M, z) - \mathcal{S}_n(M - 1, z) &= - \int_0^1 \left( \frac{d}{dt} L_n(z \ln t) \right) t^M dt - \frac{zn}{M} \\ &= M \int_0^1 L_n(z \ln t) t^{M-1} dt - L_n(0) - \frac{zn}{M} \\ &= \left(1 + \frac{z}{M}\right)^n - \frac{zn}{M} - 1. \end{aligned} \quad (6.9)$$

In the last step, we may put  $t = \exp(-v)$ , and use an extension of the Laplace transform of the Laguerre polynomial (e.g., formula 7.414.2 of [11], with  $\lambda = -z$  and  $\mu = 0$ ).

Since we have the initial value  $S_n(1, z) = -1 - nz + (1 + z)^n$ , solution of the recursion (6.7) gives the Proposition.

An alternative proof of Proposition 10 is now obvious. For from binomial expansion we obtain

$$\begin{aligned} \sum_{j=1}^M \left(1 + \frac{z}{j}\right)^n &= \sum_{j=1}^M \sum_{\ell=0}^n \binom{n}{\ell} \frac{z^\ell}{j^\ell} = \sum_{\ell=0}^n \binom{n}{\ell} z^\ell H_M^{(\ell)} \\ &= \sum_{\ell=2}^n \binom{n}{\ell} z^\ell H_M^{(\ell)} + M + nzH_M. \end{aligned} \quad (6.10)$$

It is possible to expand the factor  $(t - 1)^{-1}$  of Eqs. (6.2) and (6.4) as a geometric series, thereby giving another series representation of the sum  $\mathcal{S}_n$ , and providing a third proof of Proposition 9. We have

**Corollary 3.** We have

$$\mathcal{S}_n(M, z) = -nzH_M - M + \sum_{j=1}^{\infty} \left[ \left(1 + \frac{z}{j}\right)^n - \left(1 + \frac{z}{M+j}\right)^n \right]. \quad (6.11)$$

A simple shift of summation index here recovers the relation (6.6).

In order to obtain the result (6.11), we write

$$\mathcal{S}_n(M, z) = -z \sum_{j=0}^{\infty} \int_0^{\infty} [L_{n-1}^1(-zv) - n][e^{-(M+j+1)v} - e^{-(j+1)v}] dv. \quad (6.12)$$

We then use for  $\text{Re } k > 0$ ,

$$\int_0^{\infty} L_{n-1}^1(-zv) e^{-kv} dv = \frac{1}{z} \left[ \left(1 + \frac{z}{k}\right)^n - 1 \right]. \quad (6.13)$$

*Remarks.* Corollary 3 must be equivalent to using the partial fractions form of  $H_M^{(m)}$  coming from that for the polygamma function, in (6.1), and then interchanging summations.

By manipulating  $\mathcal{S}_n(M, z)$  we may obtain related sums. For instance, by integrating Eq. (6.1), using Proposition 8, and interchanging integrations, we have

$$\begin{aligned} \sum_{m=2}^n \binom{n}{m} \frac{H_M^{(m)}}{m+1} u^m &= u \int_0^1 \left[ \frac{L_{n-1}^2(u \ln t)}{n+1} - \frac{n}{2} \right] \left( \frac{t^M - 1}{t - 1} \right) dt \\ &= \frac{1}{u} \sum_{j=2}^n \frac{j}{(n+1)} \left[ \left( 1 + \frac{u}{j} \right)^{n+1} - 1 \right] - M - \frac{n}{2} u. \end{aligned} \quad (6.14)$$

From Proposition 10 we obtain the combination

$$\begin{aligned} \mathcal{S}_n(M, -1) - \mathcal{S}_n(M, -1/2) &= \sum_{j=1}^M \left[ \left( 1 - \frac{1}{j} \right)^n - \left( 1 - \frac{1}{2j} \right)^n \right] + \frac{n}{2} H_M \\ &= \sum_{j=1}^M \left[ \frac{n}{2} \frac{1}{j} + \left( 1 - \frac{1}{j} \right)^n - \left( 1 - \frac{1}{2j} \right)^n \right]. \end{aligned} \quad (6.15)$$

The latter expression is the  $M < \infty$  form of Eq. (19) of Ref. [9].

Plainly, the sums of this section may be generalized to those such as

$$\mathcal{S}_n(M, p, q, r, z) \equiv \sum_{m=2}^n m^p \binom{n}{m}^r [H_M^{(m)}]^q z^m. \quad (6.16)$$

### Acknowledgement

This work was supported in part by Air Force contract number FA8750-06-1-0001.



## References

- [1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards (1972).
- [2] B. C. Berndt, Ramanujan's notebooks, part I, Springer (1985).
- [3] B. C. Berndt, Ramanujan's notebooks, part IV, Springer (1994).
- [4] T. J. I'A. Bromwich, Certain potential functions and a new solution of Laplace's equation, Proc. London Math. Soc. **12**, 100-125 (1913).
- [5] W. Chu and L. De Donno, Hypergeometric series and harmonic number identities, Adv. Appl. Math. **34**, 123-137 (2005).
- [6] M. W. Coffey, On some log-cosine integrals related to  $\zeta(3)$ ,  $\zeta(4)$ , and  $\zeta(6)$ , J. Comp. Appl. Math. **159**, 205-215 (2003).
- [7] M. W. Coffey, On one-dimensional digamma and polygamma series related to the evaluation of Feynman diagrams, J. Comp. Appl. Math. **183**, 84-100 (2005).
- [8] M. W. Coffey, On a three-dimensional symmetric Ising tetrahedron, and contributions to the theory of the dilogarithm and Clausen functions, J. Math. Phys. **49**, 043510-1-32 (2008), arXiv/math-ph/08010273v2.
- [9] M. W. Coffey, Toward verification of the Riemann hypothesis: Application of the Li criterion, Math. Physics, Analysis and Geometry **8**, 211-255 (2005), arXiv/math-ph/0505052.

- [10] P. Flajolet and B. Salvy, Euler sums and contour integral representation, *Exptl. Math.* **7**, 15-35 (1998).
- [11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).
- [12] A. E. Jolliffe, A form for  $\frac{d}{dn}P_n(\mu)$ , where  $P_n(\mu)$  is the Legendre polynomial of degree  $n$ , *Mess. Math.* **49**, 125 (1919).
- [13] J. Kaucký, *Combinatorial identities*, Veda, Bratislava (1975).
- [14] A. Panholzer and H. Prodinger, Binary search tree recursions with harmonic toll functions, *J. Comp. Appl. Math.* **142**, 211-225 (2002).
- [15] P. Paule and C. Schneider, Computer proofs of a new family of harmonic number identities, *Adv. Appl. Math.* **31**, 359-378 (2003).
- [16] G. Rutledge and R. D. Douglass, Integral functions associated with certain binomial coefficient sums, *Amer. Math. Monthly*, **43**, 27-32 (1936).
- [17] R. Szmytkowski, On the derivative of the Legendre function of the first kind with respect to its degree, *J. Phys. A* **39**, 15147-15172 (2006), *J. Phys. A* **40**, 7819-7820 (2007) (E), Addendum *J. Phys. A* **40**, 14887-14891 (2007).
- [18] D. A. Zave, A series expansion involving the harmonic numbers, *Info. Proc. Lett.* **5**, 75-77 (1976).